BOOLEAN MATRICES AND GRAPH THEORY

by

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Introduction. A net is a set of points between any two of which may be a connecting line. We will consider directed nets, in which the connecting lines have a direction: if the line v_{12} connecting points p_1 and p_2 is directed from p_1 toward p_2, then p_1 is called the origin of the line, and p_2 is called the insertion of the line. In particular, we will restrict ourselves to nets in which there cannot be two or more lines in the same direction between the same pair of points, and in which a line can only connect two distinct points: such a net is called a directed graph. Figures 1 a and 1 b are not valid directed graphs; Fig. 1 c is a valid directed graph.

The study of directed graphs has many applications in information and computer science. The trees of the previous section are directed graphs; the skeletons of flow charts are graphs; the state diagrams to be studied in Part V below are related to directed graphs; and so forth. Information flow in communication network can be analyzed in terms of directed graphs, as can social-structures and many mathematical relationships. In this short section we can do no more than present a bare introduction to this subject; nevertheless, its importance in information and computer science necessitates its inclusion, however briefly, in this survey.

The Incidence Matrix G. Associated with a directed graph is a Boolean matrix G, called the incidence matrix, in which $g_{ij} = 1$ if there is a line in the graph with origin p_i and insertion p_j, and $g_{ij} = 0$ otherwise. Thus for the directed graph of Fig. 1 c we have

$$G = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Note that if we renumbered the points as in Fig. 2 , then a different matrix would result, even though the graph would have the same basic structure. For the numbering of Fig. 2 we have

$$G' = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \end{array}$$

If we wanted to show that the two matrices indeed represented the same basic graph structure, then we would have to demonstrate that there is a renumbering, or permutation, of the rows and columns of G which would make it identical to G'. Recall that when a column-unitary matrix multiplies a matrix on its left, the result is to permute the columns of that matrix. Similarly when a row-unitary matrix (with a single unit in each row) multiplies a matrix on its right, it permutes the rows of that matrix. Since we must permute† both the columns and rows of G to get G', we must multiply G both on the left and right. To get the same permutation of rows and columns, the row- and column-unitary matrices must be transposes of each other. Thus if P is the desired permutation, then

$$P^t \otimes G \otimes P = G' \tag{1}$$

For our case we see from Fig. 2 that

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \end{array}$$

whence from Eq. (1) we find

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

which is G', as desired.

† A true permutation matrix P is a (row or column) unitary matrix with a single unit in each row and each column. If point i → point j, then $P_{ij} = 1$ in P.

For our case of directed graphs (i.e. where $g_{ii} = 0$), it can be shown that, in general, given two incidence matrices G and G′, all permutations P, if any exist at all, that satisfy Eq. (1) can be generated by the following process:

Step 1. For each row j of G (written as a column) and each row i of G′ (written as a row) form†

$$S^{ij} = \begin{pmatrix} j^{th} \text{ row of G} \\ \text{written as a} \\ \text{column} \end{pmatrix} \theta \begin{pmatrix} i^{th} \text{ row of G}' \\ \text{written as a} \\ \text{row} \end{pmatrix}$$

$$= \begin{pmatrix} g_{j1} \\ g_{j2} \\ \vdots \\ g_{jJ} \end{pmatrix} \theta \; (g'_{i1} \; g'_{i2} \cdots g'_{iJ})$$

Step 2. Form every product

$$T_k = S^{1\alpha_1} \cdot S^{2\alpha_2} \cdot\cdots\cdot S^{J\alpha_J}$$

where the α_n are chosen, in some order, from 1, 2, ..., J, so that $\alpha_n \neq \alpha_m$ (i.e. no two alphas are the same in the same product). Retain only those T_k that do not have an all-zero column or row.

Step 3. For each such product T_k form the permutation matrix P_k with elements $P_{i,\alpha_i} = 1$, zeros otherwise (i.e. with a unit element corresponding to each term in the product). If

$$P_k \to T_k$$

then $P = P_k^{t}$ is a permutation that satisfies Eq. (1).

---

†If A is a matrix with a single column of elements a_i, and B is a matrix with a single row of elements b_j, then the elements of the θ (theta) product of A and B, namely $C = A \theta B$, are

$$c_{ij} = a_i b_j + \bar{a}_i \cdot \bar{b}_j$$

As an example, consider G and G′ as given above, namely

$$G = \begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&1&0&1\\ 0&1&0&0 \end{pmatrix} \qquad G' = \begin{pmatrix} 0&1&1&0\\ 0&0&1&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix}$$

Then for the S^{ij} matrices we have

$$S^{00} = \xcancel{\begin{pmatrix} 1&0&0&1\\ 1&0&0&1\\ 1&0&0&1\\ 1&0&0&1 \end{pmatrix}} \quad S^{01} = \xcancel{\begin{pmatrix} 1&1&0&1\\ 1&1&0&1\\ 1&1&0&1\\ 1&1&0&1 \end{pmatrix}} \quad S^{02} = \xcancel{\begin{pmatrix} 0&1&1&1\\ 0&1&1&1\\ 0&1&1&1\\ 0&1&1&1 \end{pmatrix}} \quad S^{03} = \begin{pmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{pmatrix}$$

$$S^{10} = \begin{pmatrix} 1&0&0&1\\ 1&0&0&1\\ 0&1&1&0\\ 1&0&0&1 \end{pmatrix} \quad S^{11} = \begin{pmatrix} 1&1&0&1\\ 1&1&0&1\\ 0&0&1&0\\ 1&1&0&1 \end{pmatrix} \quad S^{12} = \begin{pmatrix} 0&1&1&1\\ 0&1&1&1\\ 1&0&0&0\\ 0&1&1&1 \end{pmatrix} \quad S^{13} = \xcancel{\begin{pmatrix} 1&1&0&1\\ 1&1&0&1\\ 1&1&0&1\\ 1&1&0&1 \end{pmatrix}}$$

$$S^{20} = \begin{pmatrix} 1&0&0&1\\ 0&1&1&0\\ 1&0&0&1\\ 0&1&1&0 \end{pmatrix} \quad S^{21} = \begin{pmatrix} 1&1&0&1\\ 0&0&1&0\\ 1&1&0&1\\ 0&0&1&0 \end{pmatrix} \quad S^{22} = \begin{pmatrix} 0&1&1&1\\ 1&0&0&0\\ 0&1&1&1\\ 1&0&0&0 \end{pmatrix} \quad S^{23} = \xcancel{\begin{pmatrix} 1&0&1&0\\ 1&0&1&0\\ 1&0&1&0\\ 1&0&1&0 \end{pmatrix}}$$

$$S^{30} = \begin{pmatrix} 1&0&0&1\\ 0&1&1&0\\ 1&0&0&1\\ 1&0&0&1 \end{pmatrix} \quad S^{31} = \begin{pmatrix} 1&1&0&1\\ 0&0&1&0\\ 1&1&0&1\\ 1&1&0&1 \end{pmatrix} \quad S^{32} = \begin{pmatrix} 0&1&1&1\\ 1&0&0&0\\ 0&1&1&1\\ 0&1&1&1 \end{pmatrix} \quad S^{33} = \xcancel{\begin{pmatrix} 1&0&1&1\\ 1&0&1&1\\ 1&0&1&1\\ 1&0&1&1 \end{pmatrix}}$$

Note that S^{00}, S^{01}, S^{02}, S^{13}, S^{23}, and S^{33} were crossed off because they each had a zero column. For the first two rows of matrices, we can form the products

$$S^{03}.S^{10} = \begin{pmatrix} 1&0&0&1\\ 1&0&0&1\\ 0&1&1&0\\ 1&0&0&1 \end{pmatrix} \quad S^{03}.S^{11} = \begin{pmatrix} 1&1&0&1\\ 1&1&0&1\\ 0&0&1&0\\ 1&1&0&1 \end{pmatrix} \quad S^{03}.S^{12} = \begin{pmatrix} 0&1&1&1\\ 0&1&1&1\\ 1&0&0&0\\ 0&1&1&1 \end{pmatrix}$$

Including the third row of S^{ij} matrices, only

$$S^{03}.S^{10}.S^{22} = \begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&1&1&0\\ 1&0&0&0 \end{pmatrix} \quad S^{03}.S^{11}.S^{22} = \begin{pmatrix} 0&1&0&1\\ 1&0&0&0\\ 0&0&1&0\\ 1&0&0&0 \end{pmatrix} \quad S^{03}.S^{12}.S^{20} = \begin{pmatrix} 0&0&0&1\\ 0&1&1&0\\ 1&0&0&0\\ 0&1&1&0 \end{pmatrix}$$

and $S^{03}.S^{12}.S^{21} = \begin{pmatrix} 0&1&0&1\\ 0&0&1&0\\ 1&0&0&0\\ 0&0&1&0 \end{pmatrix}$

need be retained as having no all-zero columns (or rows).

Finally, including the fourth row of S^{ij} matrices, we find

$$S^{03}.S^{12}.S^{20}.S^{31} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = T_k$$

as the only product with no all-zero columns or rows. Thus

$$P_k = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

and certainly $P_k \rightarrow T_k$. Hence

$$P = P_k{}^t = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is the desired permutation.

<u>The Reachability Matrix R</u>. A <u>path</u> from P_i to P_j is a collection of points P_i, P_k, P_n, P_m, $\ldots P_r$, P_j and the lines v_{ik}, v_{kn}, v_{nm}, $\ldots$, v_{rj}, where the insertion of each line is the origin of the next, except for the last. If such a path from P_i to P_j actually exists in a graph, then point P_j is said to be <u>reachable</u> from point P_i in the graph. For instance, in Fig. 3 we see that P_3 is reachable from P_1, but P_0 is not reachable from any other point on the graph. The <u>reachability matrix</u> R of a graph has elements $r_{ij} = 1$ if P_j is reachable from P_i, and zero otherwise. Note that every point is trivially reachable from itself, and hence $r_{ii} = 1$ for all points P_i. For Fig. 3, the reachability matrix is

$$R = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \qquad \text{with } G = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

clearly $G \rightarrow R$, for if $g_{ij} = 1$, then P_j is reachable from P_i and hence $r_{ij} = 1$. For any point P_i, we can determine the set of points that can be reached from P_i

by inspecting the i^{th} row of R: the set corresponds to those columns with units in that row. Similarly we can find the set of points from which P_i is reachable by inspecting the i^{th} column of R: the set corresponds to those rows with units in that column.

The matrix R can be obtained from the matrix G of a graph as follows: Note that G gives the "one step" reachability of the graph. If we form

$$G \otimes G = G^2$$

we get the "two step" reachability of the graph, i.e. all those points that can be reached from another point in two lines. For instance, for Fig. 3 ,

$$G = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \quad \text{and } G \otimes G = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Similarly, a "three step" reachable graph can be constructed by forming

$$G \otimes G \otimes G = G^3$$

For our illustration, this becomes

$$(G \otimes G) \otimes G = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This process can be continued to find an "n step" reachable matrix G^n. Now the reachability matrix R includes all reachable steps, and the unit diagonal matrix I as well. That is,

$$R = I + G + G^2 + G^2 + \ldots + G^n$$

where I has the elements $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$. Therefore, to construct the reachability matrix R from the incidence matrix G, (logically) add to I the

matrix G and its successively higher powers until no new units can be included in R. For our case we have

$$R = I + G + G^2 + G^3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Finally, note that since $I \otimes A = A \otimes I$ we have and $G + G = G$,

$$(I + G)^2 = (I + G) \otimes (I + G) = I + G + G^2$$

or in general

$$(I + G)^n = I + G + G^2 + \dots + G^n$$

Then, if no new units can be included after the n^{th} step,

$$R = (I + G)^n$$

For our example, we have

$$(I+G) = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad (I+G)^2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad (I+G)^3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

whence $R = (I+G)^3$

Connectedness and Components. A directed graph is called strong, or strongly connected, when every point is reachable from every other point of the graph. A directed graph is weakly connected when for every two points, at least one point is reachable from the other (but not necessarily the reverse). A directed graph is unconnected when there exist at least two points neither of which is reachable from the other, but there is at least a sequence of lines, disregarding direction, between every two points. A directed graph is disconnected when it is not connected in any of the just-mentioned ways. Figure 4 illustrates the types of connectedness of graphs.

If a graph is strongly connected, then P_i must be reachable from P_j <u>and</u> P_j must be reachable from P_i, for every pair of points P_i and P_j of the graph. Hence, if J represents the matrix <u>all</u> of whose elements are units, then

$$R = J$$

if and only if the graph is strongly connected. If the graph is weakly connected, then P_i must be reachable from P_j <u>or</u> P_j must be reachable from P_i. Hence if R^t is the transpose of the reachability matrix R then

$$R + R^t = J$$

if and only if the graph is weakly connected. Of course strong connectedness implies weak connectedness. For our illustrations of Fig. 4, we have:

(a) $$R = \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

(b) $$R = \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \quad R^t = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad R + R^t = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(c) $$R = \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \quad R^t = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad R + R^t = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(d) $$R = \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \quad R^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad R + R^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Strong subcomponents of a graph can be recognized. For instance, consider Fig. 5 with the matrices

$$G = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \quad \text{and } R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Now if P_j is reachable from P_i, then $r_{ij} = 1$; if P_i is reachable from P_j then $r^t_{ij} = r_{ji} = 1$. Hence the unit elements of the <u>product</u> $R \cdot R^t$ represent mutually

reachable pairs of points which are therefore members of some strong subcomponent of the graph. For our illustration of Fig. 5 , we have

$$
R \cdot R^t = \begin{pmatrix} 1&1&1&1&1&0 \\ 1&1&1&1&1&0 \\ 0&0&1&1&1&0 \\ 0&0&1&1&1&0 \\ 0&0&1&1&1&0 \\ 0&0&1&1&1&1 \end{pmatrix} \cdot \begin{pmatrix} 1&1&0&0&0&0 \\ 1&1&0&0&0&0 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 0&0&0&0&0&1 \end{pmatrix} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc:ccc:c} 0&1&2&3&4&5 \end{array} \\ \left(\begin{array}{cc:ccc:c} 1&1&0&0&0&0 \\ 1&1&0&0&0&0 \\ \hdashline 0&0&1&1&1&0 \\ 0&0&1&1&1&0 \\ 0&0&1&1&1&0 \\ \hdashline 0&0&0&0&0&1 \end{array}\right) \end{array}
$$

Here the dashed lines represent a decomposition of the matrix into the strong subcomponents of the graph, namely point sets $\{P_0, P_1\}$, $\{P_2, P_3, P_4\}$, and $\{P_5\}$. Note that a single point is always a strong subcomponent of itself (why?). The complete decomposition would be characterized by

$\{P_0, P_1\}$: $\quad G = \begin{pmatrix} 0&1 \\ 1&0 \end{pmatrix} \quad R = \begin{pmatrix} 1&1 \\ 1&1 \end{pmatrix}$

$\{P_2, P_3, P_4\}$: $\quad G = \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix} \quad R = \begin{pmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{pmatrix}$

$\{P_5\}$: $\quad G = (1) \quad R = (1)$

We can construct a new directed graph, called the <u>condensed graph</u>, that displays the relationships of the strong components (see Fig. 6). The condensed graph will have points corresponding to the strong components, say Q_1, Q_2, and Q_3 for $\{P_0, P_1\}$, $\{P_2, P_3, P_4\}$ and $\{P_5\}$. If at least one line connects a point of one component with a point of another, then there will be a line in the same direction connecting the corresponding points in the condensed graph. For our illustration, from the full-sized G above (or see Fig. 5) we will have a line V_{01} connecting Q_0 and Q_1 and a line v_{21} connecting Q_2 and Q_1 (see Fig. 6). For our condensed graph, we now have

$$
G = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0&1&2 \end{array} \\ \begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&1&0 \end{pmatrix} \end{array} \qquad R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0&1&2 \end{array} \\ \begin{pmatrix} 1&1&0 \\ 0&1&0 \\ 0&1&1 \end{pmatrix} \end{array} \qquad R \cdot R^t = \begin{pmatrix} 1&1&0 \\ 1&1&1 \\ 0&1&1 \end{pmatrix}
$$

Of course the condensed graph is not strongly connected (why?) and for our example is not weakly connected. Another very important question can be answered
condensed graph, namely, what is the minimum set of points of the original
from our graph from which all other points can be reached. Clearly, if we find such a minimum set for the condensed graph, then we have answered the question for the original graph, since a point chosen for the condensed-graph case can be replaced by any one of the points of the strong component it represents. To find the minimum set for the condensed graph, we chose those points which correspond to columns with a single unit in the R matrix. We know, of course, that such columns (with a single unit) must exist in the condensed graph (why?). This single unit simply means that the point is reachable from itself. All remaining columns (with more than one unit) will then correspond to points reachable from the other points. For instance, for our example column zero and column two of the condensed R have single units, corresponding to points Q_0 and Q_2 (see Fig. 6). Hence a minimum set of points for the original graph (see Fig. 5) could be P_1 (from Q_0) and P_5 (from Q_2) and all points of the graph can be reached from one of these. Such a minimum set of points is called a point basis for the graph.

Application of Arithmetic Matrices to Graph Theory. Up to now in our treatment of graph theory, we have been utilizing only Boolean matrices with Boolean, or logical, operations. In this paragraph, we will turn to the use of arithmetic matrices using ordinary arithmetic operations in matrix multiplication. We will, however, again start with the incidence matrix. Our first observation is that the value of an element $g_{ij}^{(n)}$ of G^n, the n^{th} arithmetic power of the incidence matrix G, is the number of paths from P_i to P_j of length n (where by the length of a path we mean the number of lines or steps required). For instance, for the graph of Fig. 7

$$G = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad G^2 = \begin{pmatrix} 1 & 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad G^3 = \begin{pmatrix} 0 & 3 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad G^4 = \begin{pmatrix} 1 & 4 & 0 & 4 & 5 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 4 & 1 & 4 & 4 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

For G^2, note, for example, that there are two paths of length 2 from P_0 to P_3, one through P_4 and the other through P_2. For G^3, note that there are three paths of length 3 from P_0 to P_1 (namely $P_0P_3P_4P_1$, $P_0P_1P_4P_1$, and $P_0P_2P_0P_1$), and so forth. For G^4 there are five paths of length 4 from P_1 to P_4 (namely $P_0P_2P_0P_3P_4$, $P_0P_2P_0P_1P_4$, $P_0P_2P_1P_3P_4$, $P_0P_1P_4P_1P_4$, and $P_0P_3P_4P_1P_4$).

If we form the matrix $T = G \times G^t$ then

$$t_{ij} = g_{i1}\, g_{j1} + g_{i2}\, g_{j2} + \dots + g_{in}\, g_{jn}$$

Hence if both g_{ik} and g_{jk} are units, a unit will be contributed to the sum composing t_{ij}. Elements g_{ik} and g_{jk} both being units means that lines go from both P_i and P_j to P_k. Hence the value of t_{ij} is the number of points that are insertions for lines having origins at <u>both</u> P_i and P_j. For diagonal elements of T, namely t_{ii}, the value is the number of points that are insertions for lines having P_i as origin, i.e. the number of lines emanating from P_i. For the illustration of Fig. 7 , we have

$$T = G \times G^t = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Thus, for example, there are three lines emanating from (having their origins at) P_0, two from P_1, three from P_2, and one each from P_3 and P_4. By similar reasoning, we can see that for matrix $H = G^t \times G$, the value of h_{ij} is the number of points that are origins for lines having insertions at both P_i and P_j, and that value of the diagonal elements h_{ii} is the number of lines converging on P_i. For our illustration,

$$H = G^t \times G = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Thus P_3, for instance, is the point of insertion of three lines, and so forth.

Finally, let us observe that we can easily construct a distance matrix D for a graph by observation of the sequence of Boolean matrices G, G^2, G^3, ..., G^n . The value of d_{ij} is found by examining the sequence of ij^{th} elements of the matrix sequence; the exponent of the matrix in which this element first becomes a unit is the value of d_{ij}. We must, however, use the conventions that a point and itself the distance between is 0, and between two points for which there is no connecting path the distance is ∞. For instance, for Fig. 8 we have

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G^3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G^4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The distance matrix becomes

$$D = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ \infty & \infty & \infty & 0 \end{pmatrix}$$

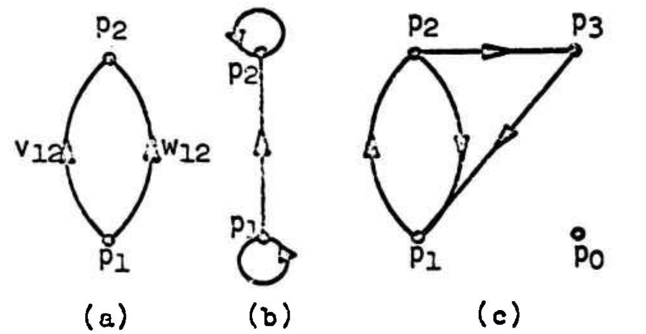

Fig. 1. (a) <u>Not</u> a valid graph because the two lines v_{12} and w_{12} are in the same direction between the same points. (b) <u>Not</u> a valid graph because loops at p_1 and p_2 do not connect distinct points. (c) <u>Valid</u> directed graph.

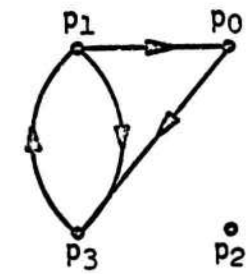

Fig. 2. A renumbering of the points of the graph of Fig. 1c, where $0 \to 2$, $1 \to 3$, $2 \to 1$, and $3 \to 0$.

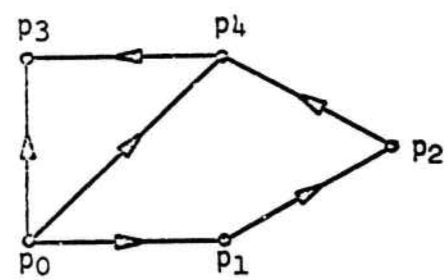

Fig. 3. Graph illustrating reactability.

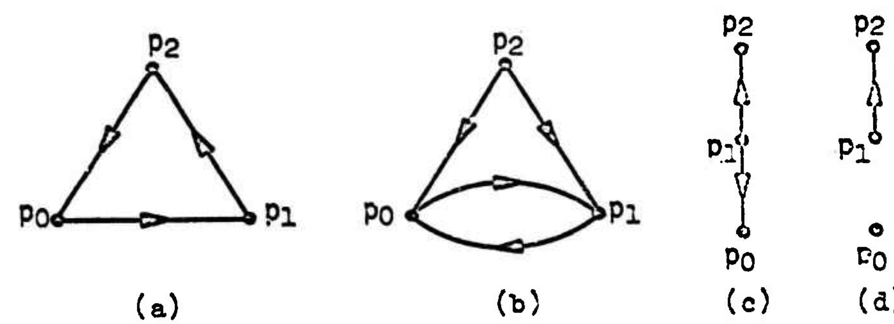

Fig. 4. (a) Strongly connected graph. (b) Weakly connected graph. (c) Unconnected graph. (d) Disconnected graph.

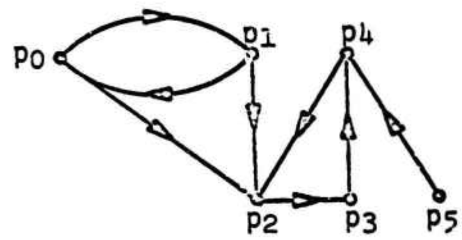

Fig. 5. Illustration of decomposition of a graph into strong subcomponents $\{p_0,p_1\}$, $\{p_2,p_3,p_4\}$, and $\{p_5\}$.

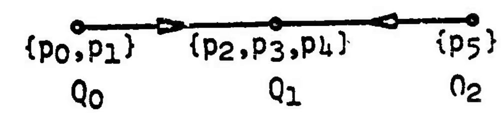

Fig. 6. The condensed graph corresponding to the strong components of Fig. 5, with the new points called Q_0, Q_1, and Q_2.

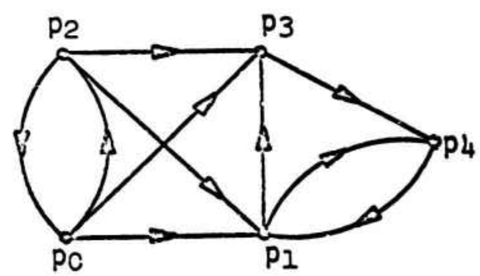

Fig. 7. Illustration for the application of arithmetic matrices.

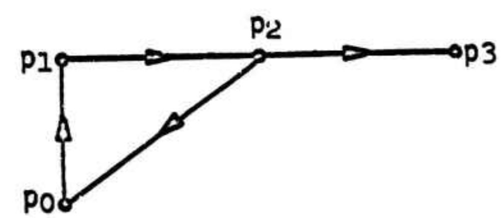

Fig. 8. Illustration for the distance matrix.